Almost Periodic Functions and Representations of the Free Group on Two Generators

Stefanie Wang

Iowa State University
Department of Mathematics

Workshop in Noncommutative Analysis
June 2016

Quasigroups

Definition

A *quasigroup* $(Q, \cdot, /, \backslash)$ is an algebra with three binary operations, multiplication $\cdot$, right division $/$, and left division $\backslash$ such that for all $x, y \in Q$,

$$y\backslash(y \cdot x) = x = (x \cdot y)/y \quad (1)$$

$$y \cdot (y\backslash x) = x = (x/y) \cdot y \quad (2)$$

are satisfied.

Definition

A *pique* is a quasigroup $(Q, \cdot, /, \backslash)$ with a pointed idempotent element e such that $e \cdot e = e$.

Linear Piques

Definition

A quasigroup $(A, \cdot, /, \backslash)$ is said to be $\mathbb{Z}$-linear if there is a unital $\mathbb{Z}$-module structure $(A, +, 0)$, with automorphisms λ and ρ such that

$$x \cdot y = x^\rho + y^\lambda, \;\; x/y = (x - y^\lambda)^{\rho^{-1}}, \text{and } x\backslash y = (y - x^\rho)^{\lambda^{-1}} \qquad (3)$$

for $x, y \in A$.

Examples of piques:

- $(\mathbb{Z}/4, x \circ_1 y = x(1\ 2\ 3) + y(1\ 2))$
  $2 \circ_1 1 = 2(1\ 2\ 3) + 1(1\ 2) = 3 + 2 = 5 \equiv 1 \mod 4$
- ($\mathbb{Z}$-linear) $(\mathbb{Z}/4, x \circ_2 y = x(1\ 3) + y(1\ 3))$
- ($\mathbb{Z}$-linear) Integers under subtraction

Representations of $\mathbb{Z}$-Linear Piques

Definition

Let $(A, \cdot, /, \backslash)$ be a $\mathbb{Z}$-linear pique such that $x \cdot y = x\rho + y\lambda$. Let $\langle R, L \rangle$ be the free group on two generators. A *$\mathbb{Z}$-linear representation of* $(A, \cdot, /, \backslash)$ is a homomorphism $\alpha : \langle R, L \rangle \to \mathrm{Aut}(A, +, 0)$.

Definition

Let $\alpha : \langle R, L \rangle \to \mathrm{Aut}(A, +, 0)$ and $\beta : \langle R, L \rangle \to \mathrm{Aut}(B, +, 0)$ be $\mathbb{Z}$-linear representations. We say α, β are *isomorphic representations* if there exists a $\mathbb{Z}$-module isomorphism $f : A \to B$ such that for all $g \in \langle R, L \rangle$, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g^{\alpha}} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g^{\beta}} & B \end{array}$$

Isomorphism of $\mathbb{Z}$-Linear Piques

Theorem (Isomorphism of $\mathbb{Z}$-Linear Piques)

Let $(A, \circ_1)$ and $(B, \circ_2)$ be two $\mathbb{Z}$-linear piques. The piques are isomorphic if and only if there exists a pair of equivalent representations of each pique.

Proof

Suppose $f : (A, \circ_1) \to (B, \circ_2)$ is a pique isomorphism, i.e. for all $x, y \in A$, $(x \circ_1 y)f = xf \circ_2 yf$. Let $\alpha : \langle R, L \rangle \to \mathrm{Aut}(A, +, 0)$ and $\beta : \langle R, L \rangle \to \mathrm{Aut}(B, +, 0)$ be $\mathbb{Z}$-linear representations. Observe

$$aR^\alpha f = (a \circ_1 0)^f = a^f \circ_2 0 = a^f R^\beta, \tag{4}$$

$$aL^\alpha f = (0 \circ_1 a)^f = 0 \circ_2 a^f = a^f L^\beta. \tag{5}$$

Then for all $a \in A$, $g \in \langle R, L \rangle$, $ag^\alpha f = afg^\beta$ holds.

Isomorphism of $\mathbb{Z}$-Linear Piques

Proof (cont.)

Conversely, suppose $\alpha : \langle R, L\rangle \to \mathrm{Aut}(A, +, 0)$ and $\beta : \langle R, L\rangle \to \mathrm{Aut}(B, +, 0)$ are isomorphic $\mathbb{Z}$-linear representations.

- Let $f : A \to B$ be the intertwining, i.e. for all $a \in A, g \in \langle R, L\rangle$, $ag^\alpha f = afg^\beta$.
- Need to verify $x, y \in A$, $(x \circ_1 y)f = xf \circ_2 yf$.
- Then

$$
\begin{aligned}
(x \circ_1 y)^f &= (xR^\alpha + yL^\alpha)f && (6)\\
&= (xR^\alpha)f + (yL^\alpha)f && (7)\\
&= x^f R^\beta + y^f L^\beta && (8)\\
&= x^f \circ_2 y^f && (9)
\end{aligned}
$$

□

Multiplication Groups

Recall from the quasigroup definition

$$y \cdot (y \backslash x) = x = (x/y) \cdot y.$$

Definition

For a quasigroup $(Q, \cdot, /, \backslash)$, one has *right multiplication* $R_Q(q)$ or $R_.(q)$ defined as

$$R(q) : Q \to Q; x \mapsto x \cdot q \tag{10}$$

and *left multiplication* $L_Q(q)$ or $L_.(q)$ defined as

$$L(q) : Q \to Q; x \mapsto q \cdot x. \tag{11}$$

Universal Stabilizer

Cayley graph of Q is defined to be the labeled directed graph with vertex set Q. For $(x, y) \in Q \times Q$, there are two directed edges,

$$R(x \searrow y) := \langle x, R(x\backslash y), y \rangle \text{ or } x \xrightarrow{R(x\backslash y)} y \tag{12}$$

and

$$L(x \swarrow y) := \langle x, L(y/x), y \rangle \text{ or } y \xleftarrow{L(y/x)} x. \tag{13}$$

Remark

Paths in the Cayley graph are words in the universal multiplication group.

Definition

Let Q be a quasigroup with fixed element e. The *universal stabilizer* $\tilde{G}_e$ *of* $e \in Q$ in the category $\mathbf{Q}$ is the free group of loops based at the vertex e.

Bohr compactification

Let Q be a quasigroup with fixed element e.
Equip $\tilde{G}_e$ with discrete topology.
Bohr compactification of $\tilde{G}_e$ is $K = K(Q)$.

Sketch of Bohr compactification:

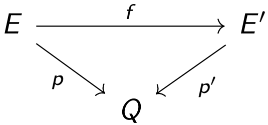
- $U(V_i)$ - unitary groups of finite-dimensional complex inner product spaces V_i.
- Take $\{\alpha_i : \tilde{G}_e \to U(V_i) | i \in I\}$ of representatives for the equivalence classes of continuous representations $\alpha_i : \tilde{G}_e \to U(V_i)$
- $\prod_{i \in I} \alpha_i : \tilde{G}_e \to \prod_{i \in I} U(V_i)$ and $K = \overline{(\tilde{G}_e)\prod_{i \in I} \alpha_i}$
- For $i \in I$, $U(V_i)$ is closed and bounded, so $\prod_{i \in I} U(V_i)$ is compact.
- K a closed subset, so K is compact

Quasigroup Modules

Definition

For a quasigroup Q, a *Q-module* is an abelian group object $p : E \to Q$ in the category $\mathbf{Q}/Q$ of quasigroups over Q. Objects in this categories are morphisms whose codomain is Q.

Morphisms in this category:



If $g : E' \to E$ is such that $fg = \mathrm{id}_{p:E\to Q}$ and $gf = \mathrm{id}_{p':E\to Q}$, then p, p' isomorphic.

Ordinary Q-modules

Ordinary Q-modules are finite-dimensional complex vector spaces.
Take $e \in Q$.
$V = p^{-1}\{e\}$ is a finite-dimensional unitary $\tilde{G}_e$-module.
Corresponds to a finite-dimensional continuous unitary representation $\sigma_E : K \to U(V)$.

Theorem (Theorem 12.3, Smith)

Each ordinary Q-module $p : E \to Q$ with $V = p^{-1}\{e\}$ is determined (up to equivalence) by the corresponding finite-dimensional continuous unitary representation $\sigma_E : K \to U(V)$ of the Bohr compactification $K = K(Q)$ of $\tilde{G}_e$.

Almost-periodic Functions

Definition

Let $p : E \to Q$ be an ordinary Q-module, and $\sigma_E : K \to U(p^{-1}\{e\})$ the corresponding finite-dimensional unitary continuous representation of the Bohr compactification $K = K(Q)$ of $\tilde{G}_e$, as in Theorem 12.3. Then the *analytical character* χ_E of E is the restriction to $\tilde{G}_e$ of the character $\chi_{\sigma_E} : K \to \mathbb{C}; x \mapsto \mathrm{Tr}_{\sigma_E}(x)$ of σ_E.

Let $f : K \to \mathbb{C}$ be a continuous function. The restriction of f to $\tilde{G}_e$ is known as an *almost-periodic function* on $\tilde{G}_e$.

Almost-periodic Functions

Theorem (Theorem 12.4, Smith)

Let E be an ordinary Q-module. Then E is classified up to equivalence by its analytical character χ_E, which is an almost-periodic function on the universal stabilizer $\tilde{G}_e$ of e in Q.

We will consider Q as the singleton $\{e\}$.

Constructing Character Tables

Consider the pique $(\mathbb{Z}/4, x(1\ 3) + y(1\ 3))$. Let $\alpha : \langle R, L \rangle \to \mathrm{Aut}(\mathbb{Z}/4, +, 0)$ be a representation. Observe:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & 1 \end{bmatrix}$$

Set $L^{\alpha},\ R^{\alpha} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, so $\chi_{\alpha}(R) = \chi_{\alpha}(L) := 2$.

Remark

Let $(A, \cdot, /, \backslash)$. For $g \in \langle R, L \rangle$ and $\alpha : \langle R, L \rangle \to \mathrm{Aut}(A, +, 0)$, $\chi(g)$ corresponds to the number of fixed points of the permutation for ρ, λ.

A Curious Example on $\mathbb{Z}/5$ Piques

$\mathrm{Aut}(\mathbb{Z}/5) \simeq (\mathbb{Z}/5)^\times = \{1, 2, 3, 4\}$

Unit	1	2	3	4
Permutation	(1)	(1 2 4 3)	(1 3 4 2)	(1 4)(2 3)

Look at $(\mathbb{Z}/5, x \circ_1 y = x + 2y)$ and $(\mathbb{Z}/5, x \circ_2 y = x + 3y)$

$x \cdot y =$	$R \mapsto$	$L \mapsto$	$\chi(R)$	$\chi(L)$	$\chi(L^2)$	$\chi(L^3)$
$x + 2y$	(1)	(1 2 4 3)	5	1	1	1
$x + 3y$	(1)	(1 3 4 2)	5	1	1	1

For all $x, y \in \mathbb{Z}/5$,

$$(x \circ_1 x) \circ_1 x = (y \circ_1 y) \circ_1 y \tag{14}$$

$$(x \circ_2 x) \circ_2 x \neq (y \circ_2 y) \circ_2 y \tag{15}$$

Linear Piques on $\mathbb{Z}/5$

Other cases:
$\{2x+2y, 3x+3y\}, \{2x+y, 3x+y\}, \{4x+2y, 4x+3y\}, \{2x+4y, 3x+4y\}$.

Lemma

If two linear piques $(\mathbb{Z}/5, \circ_1)$ and $(\mathbb{Z}/5, \circ_2)$ have the same ordinary character, the corresponding ordinary representations of $\langle R, L\rangle$ are similar via a permutation matrix P.

$$(2\ 3)(1\ 3\ 4\ 2)(2\ 3) = (1\ 2\ 4\ 3)$$
$$(2\ 3)(1\ 4)(2\ 3)(2\ 3) = (1\ 4)(2\ 3)$$

What other linear piques have this property?

Let $n = p^k$ where p is an odd prime and $k \in \mathbb{Z}^+$.

Lemma

If two linear piques defined on $\mathbb{Z}/n$ have the same ordinary character for representations α_1, α_2 of $\langle R, L \rangle$, then ρ, λ in the respective piques have the same cycle type.

Lemma

Let λ_1, λ_2 represent left multiplication by 0 in two linear piques $(\mathbb{Z}/n, \circ_1), (\mathbb{Z}/n, \circ_2)$. If λ_1 and λ_2 are nontrivial elements and there exists $b \in \mathrm{Aut}(\mathbb{Z}/n)$ that conjugates them, then $\lambda_1 = \lambda_2$.

Proof.

Recall $\mathrm{Aut}(\mathbb{Z}/n) \cong (\mathbb{Z}/n)^\times$. Suppose $b \in (\mathbb{Z}/n)^\times$ such that $b\lambda_1 b^{-1} = \lambda_2$. Since $(\mathbb{Z}/n)^\times$ is abelian, it follows $\lambda_1 = \lambda_2$. □

Isomorphism of Ordinary Representations of $\langle R, L\rangle$

Lemma

Let $(\mathbb{Z}/n, \circ_1), (\mathbb{Z}/n, \circ_2)$ be two linear piques with isomorphic ordinary representations α_1, α_2 of $\langle R, L\rangle$ such that L and R have the same cycle type in $\mathrm{Aut}(\mathbb{Z}/n)$. Then the ordinary representations are similar by a permutation matrix.

Theorem

Let $(\mathbb{Z}/n, \circ_1)$ and $(\mathbb{Z}/n, \circ_2)$ be two $\mathbb{Z}$-linear piques. If they yield equivalent ordinary representations α_1, α_2 of $\langle R, L\rangle$, then α_1 and α_2 are similar by a permutation matrix.

Modules over Arbitrary Unital Rings

Definition

Let S be a unital ring. Let A be a right module over S. A quasigroup $(A, \cdot, /, \backslash)$ is said to be S-linear if there is a unital S-module structure $(A, +, 0)$, with automorphisms λ and ρ such that

$$x \cdot y = x^{\rho} + y^{\lambda},\ x/y = (x - y^{\lambda})^{\rho^{-1}}, \text{and } x\backslash y = (y - x^{\rho})^{\lambda^{-1}} \tag{16}$$

for $x, y \in A$.

Theorem

Let $(A, \circ_1)$ and $(B, \circ_2)$ be two S-linear piques. The piques are isomorphic if and only if there exists a pair of equivalent representations of the free group on two generators $\alpha_1, \alpha_2 : \langle R, L\rangle \to \mathrm{Aut}(A, +, 0)$.

Reference

Smith, Jonathan DH. *An Introduction to Quasigroups and Their Representations.* 2006.

Thank you!